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On two-connected subgraph polytopes

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Abstract

We further study some known families of valid inequalities for the 2-edge-connected and 2-node-connected subgraph polytopes. For the 2-edge-connected case, we show that the odd wheel inequalities together with the obvious constraints give a complete description of the polytope for Halin graphs. For 2-node-connected subgraphs, we show that the inequalities above, plus the partition inequalities, describe the polytope for the same class of graphs.

1. Introduction

The problem of finding a 2-connected subgraph of minimum weight arises in the design of communication and transportation networks. In order to use linear programming techniques one needs a system of inequalities that defines or approximates the convex hull of incidence vectors of 2-connected subgraphs. The case when the edge weights satisfy the triangular inequality was studied in [14]. For the 2-edge-connected case a family of facets was given in [13], and it was proved that for series–parallel graphs the polytope has a simple description. The polytope of 2-node-connected subgraphs of graphs with no W_4 minor was characterized in [6]. Several classes of facets have been given in [11,9], for a more general model, and computational experience with them has been presented in [10]. The 2-edge connected case in directed graphs was studied in [3], and it was shown that facets for undirected case can be obtained by projection. The k-edge connected case was studied in [4], when multiple copies of an edge may be used; The polytope for outerplanar graphs, when k is odd, was characterized in this paper.

Proving that some constraints define facets, and showing computational experience are ways to validate these classes of inequalities. Another way to validate a family of

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inequalities is to show that although in general they only approximate the polytope, they give a complete description for simple classes of graphs. We are going to follow this path with the *odd wheel* and *partition* inequalities. We shall show that when combined with the obvious inequalities, they define the polytope for the class of Halin Graphs. This seems to indicate that for sparse graphs, these constraints are going to be useful. Halin graphs can be decomposed by 3-edge cuts, we use this property in a similar manner as Cornuéjols et al. [5] did for the Traveling Salesman Problem.

Given a graph $G = (V, E)$, a spanning subgraph $H = (V, F)$ is called *k-edge-connected* (resp. *k-node-connected*) if there are k edge-disjoint (resp. internally node-disjoint) paths between any two nodes. For $S \subset V$ we denote by $\delta(S)$ the set edges with exactly one endnode in S. For $x \in \Re^E$ and $T \subseteq E$ we abbreviate $\sum\{x(e) \mid e \in T\}$ by $x(T)$.

For a full-dimensional polyhedron P, if two inequalities $ax \geqslant \alpha$ and $bx \geqslant \beta$ define the same facet then $a = \lambda b$, $\alpha = \lambda\beta$ for $\lambda > 0$. Also if $Ax \geqslant b$ is a minimal system that defines P then there is a natural bijection between the inequalities in this set and the facets of P. So if an inequality defines a facet of P, it will appear (up to multiplication by a positive number) in any system that defines P.

Given $F \subseteq E$ the incidence vector of F is denoted by x^F. We are going to study the *2-edge-connected subgraph polytope*

$$\mathrm{TECP}(G) = \mathrm{conv}\{x^F \mid (V, F) \text{ is a 2-edge-connected subgraph of } G\},$$

and the *2-node-connected subgraph polytope*

$$\mathrm{TNCP}(G) = \mathrm{conv}\{x^F \mid (V, F) \text{ is a 2-node-connected subgraph of } G\}.$$

The traveling salesman polytope is a face of both polytopes, this suggests that finding a complete description by a system of inequalities is unlikely for general graphs. Clearly $\mathrm{TNCP}(G) \subseteq \mathrm{TECP}(G)$, so let us first concentrate in $\mathrm{TECP}(G)$. The *bound* inequalities

$$0 \leqslant x(e) \leqslant 1 \quad \text{for } e \in E$$

and the *cut* inequalities

$$x(\delta(S)) \geqslant 2 \quad \text{for } \emptyset \neq S \subset V$$

are valid for $\mathrm{TECP}(G)$. In [13] it was proved that these inequalities define $\mathrm{TECP}(G)$ if G is series–parallel. It was also proved that if G is 3-edge-connected then this is a full-dimensional polytope, $x(e) \geqslant 0$ defines a facet if e is not in a 2- or 3-edge cut, $x(e) \leqslant 1$ defines a facet if e is not in a 2-edge-cut, and a cut inequality defines a facet if the cut has at least three edges and both shores are 2-edge-connected.

Also in [13], a family of valid inequalities was introduced as follows. Consider a partition of V into $\bar{V}, V_1, \ldots, V_p$, and let $F \subseteq \delta(\bar{V})$ with $|F| = 2k + 1$, let

$$\delta(V_1, \ldots, V_p) = \bigcup_{i=1}^{p} \delta(V_i),$$

if we add the inequalities

$$x(\delta(V_i)) \geqslant 2, \quad 1 \leqslant i \leqslant p, \qquad -x(e) \geqslant -1, \quad e \in F, \qquad x(e) \geqslant 0, \quad e \in \delta(\bar{V}) \setminus F,$$

we obtain

$$2x(\Delta) \geqslant 2p - 2k - 1,$$

where $\Delta = \delta(V_1, \ldots, V_p) \setminus F$, dividing by 2 and rounding up the right-hand side we obtain

$$x(\Delta) \geqslant p - k. \tag{1.1}$$

We are going to prove that bound, cut and inequalities (1.1) define TECP(G) if G is a Halin graph.

Consider now the 2-node-connected case, all the inequalities above are valid for TNCP(G) but we need some new constraints. For this let us first notice that if a graph is 2-node-connected then when we delete any node the remainder is connected, i.e., it contains a spanning tree. The dominant of the spanning tree polytope of a graph $H = (U, F)$ is defined by

$$x(\delta(U_1, \ldots, U_p)) \geqslant p - 1 \quad \text{for every partition } U_1, \ldots, U_p \text{ of } U, \tag{1.2}$$

$$x \geqslant 0,$$

see [15, 2]. Thus we can delete any node $u \in G$ and inequalities (1.2) for $G \setminus u$, called *partition inequalities*, are valid for TNCP(G). We shall prove that bound, cut, (1.1) and partition inequalities are sufficient to define TNCP(G) if G is a Halin graph.

In order to use these inequalities in a cutting plane algorithm one needs an efficient way to find one of them that is violated. The separation problem for the cut inequalities can be solved as a sequence of minimum cut problems. Inequalities (1.1) reduce some blossom inequalities for b-matching if the sets $\{V_i\}$ are singletons, so in this case one can solve the separation problem with the procedure of [16]. It would be very interesting to have a polynomial algorithm for inequalities (1.1) in general. The separation problem for (1.2) can be solved as a sequence of $|V| \cdot |E|$ minimum cut problems using an algorithm of [7] or as a sequence of $|V|^2$ minimum cut problems using an algorithm of [1].

Before concluding this introduction we give a sufficient condition for inequalities (1.1) to be facet inducing. This will be used in Section 3.

Theorem 1.1 (Mahjoub [13]). *Let $G = (V, E)$ be a 3-edge-connected graph whose node set can be partitioned into $\bar{V}$, V_i^j, $i = 0, \ldots, 2k$, $j = 0, 1, \ldots, p_i$, so that:*

(1) *the subgraph induced by each member of the partition is 3-edge-connected;*

(2) *there is at least one edge between V_i^0 and V_{i+1}^0 for $i = 0, \ldots, 2k$ (modulo $2k+1$);*

(3) *if $p_i > 0$ there is exactly one edge between V_i^j and V_i^{j+1}, $j = 0, \ldots, p_i$; $i = 0, \ldots, 2k$, where $V_i^{p_i+1} = \bar{V}$;*

(4) *if* V_i^0, $i = 0, \ldots, 2k$, *are removed the only edges between the members of the partition that remain are among those described in* (3);

(5) *there is no edge between* V_i^0 *and* V_i^j, $j = 0, \ldots, p_i$; $i = 0, \ldots, 2k$.

Let $r_i \leqslant p_i$ *be the largest integer such that* $|\delta(V_i^{r_i})| \geqslant 3$, *and let* ε_i *be the edge between* $V_i^{r_i}$ *and* $V_i^{r_i+1}$, $0 \leqslant i \leqslant 2k$. *Set* $F = \{\varepsilon_i, 0 \leqslant i \leqslant 2k\}$, *and*

$$\Delta = \bigcup_{\substack{i=0,\ldots,2k \\ j=0,\ldots,r_i}} \delta(V_i^j)\backslash F,$$

then the inequality

$$x(\Delta) \geqslant k + 1 + \sum_{i=0}^{2k} r_i, \tag{1.3}$$

defines a facet of TECP(G).

Constraints (1.3) are called *odd wheel inequalities.*

2. Halin graphs

A *Halin* graph $G = (V, T \cup C)$ consists of a tree T that has no degree-two nodes, together with a simple cycle C whose nodes are the pendant nodes of T, the graph should be embeddable in the plane with C as the exterior face. These are examples of minimally 3-connected graphs given by Halin [12]. Any edge $e \in T$ is a unique 3-edge cut that contains two edges of C, we denote this cut by δ_e. All results in this section are valid for TECP(G) and for TNCP(G). We are going to use P(G) to denote either one of these polytopes.

Wheels are those Halin graphs with T being a star. If a Halin graph $G = (V, T \cup C)$ is not a wheel then for any nonpendant edge $e \in T$ the cut δ_e is non trivial, i.e., $\delta_e = \delta(S)$ with $|S| \geqslant 2 \leqslant |V\backslash S|$. Let G_1 be the graph obtained by shrinking S to a single node and let G_2 be obtained from G by shrinking $V\backslash S$, then G_1 and G_2 are also Halin graphs. If we keep applying this procedure recursively we are left at the end with a set of wheels. We need the following that is an adaptation of a theorem of [5].

Theorem 2.1. *Let* $G = (V, E)$ *be a graph that has a 3-edge cut* $\delta(S)$. *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be obtained from* G *by shrinking* S *and* $V\backslash S$ *respectively. Then a system of linear inequalities sufficient to define* P(G) *is obtained from the union of the systems that define* P(G_1) *and* P(G_2), *and by identifying the variables associated with the edges in* $\delta(S)$.

Proof. Let Q be the polytope defined by the union of these systems. Clearly P(G) $\subseteq Q$, so we have to prove that every vector $x \in Q$ is a convex combination of vectors in P(G).

Assume that e, f and g are the edges in $G_1 \cap G_2$. The restriction x^1 of x to the component set E_1 belongs to $\mathrm{P}(G_1)$ thus

$$x^1 = \sum_{i \in I} \lambda_i y^i, \quad \text{with} \sum_{i \in I} \lambda_i = 1, \ \lambda \geqslant 0,$$

and the vectors $\{y^i\}$ are extreme points of $\mathrm{P}(G_1)$.

Let

$$l_{ef} = \sum \{\lambda_i \colon i \in I \text{ such that } y^i(e) = y^i(f) = 1, \ y^i(g) = 0\},$$

$$l_{fg} = \sum \{\lambda_i \colon i \in I \text{ such that } y^i(f) = y^i(g) = 1, \ y^i(e) = 0\},$$

$$l_{eg} = \sum \{\lambda_i \colon i \in I \text{ such that } y^i(e) = y^i(g) = 1, \ y^i(f) = 0\},$$

$$l_{efg} = \sum \{\lambda_i \colon i \in I \text{ such that } y^i(e) = y^i(f) = y^i(g) = 1\}.$$

Note that

$$l_{ef} + l_{eg} + l_{efg} = x(e), \qquad l_{ef} + l_{fg} + l_{efg} = x(f),$$

$$l_{eg} + l_{fg} + l_{efg} = x(g), \qquad l_{ef} + l_{eg} + l_{fg} + l_{efg} = 1.$$

This uniquely determines l_{ef}, l_{eg}, l_{fg} and l_{efg}, given x.

Similarly, for the restriction x^2 of x to E_2, we have

$$x^2 = \sum_{j \in J} \mu_j z^j, \quad \text{with} \sum_{j \in J} \mu_j = 1, \ \mu \geqslant 0,$$

where the vectors $\{z^j\}$ are extreme points of $\mathrm{P}(G_2)$.

Let

$$m_{ef} = \sum \{\mu_j \colon j \in J \text{ such that } z^j(e) = z^j(f) = 1, \ z^j(g) = 0\},$$

$$m_{fg} = \sum \{\mu_j \colon j \in J \text{ such that } z^j(f) = z^j(g) = 1, \ z^j(e) = 0\},$$

$$m_{eg} = \sum \{\mu_j \colon j \in J \text{ such that } z^j(e) = z^j(g) = 1, \ z_i(f) = 0\},$$

$$m_{efg} = \sum \{\mu_j \colon j \in J \text{ such that } z^j(e) = z^j(f) = z^j(g) = 1\}.$$

Then

$$m_{ef} + m_{eg} + m_{efg} = x(e), \qquad m_{ef} + m_{fg} + m_{efg} = x(f),$$

$$m_{eg} + m_{fg} + m_{efg} = x(g), \qquad m_{ef} + m_{eg} + m_{fg} + m_{efg} = 1.$$

This system of equations has a unique solution, then $l_{ef} = m_{ef}$, $l_{fg} = m_{fg}$, $l_{eg} = m_{eg}$ and $l_{efg} = m_{efg}$. Thus we can match vectors y^i with vectors z^j to form incidence vectors of 2-edge-connected subgraphs of G, say $\{\chi^p\}$, and a family of coefficients $\{\beta_p\}$ such that

$$x = \sum \beta_p \chi^p, \quad \sum \beta_p = 1, \quad \text{and} \quad \beta \geqslant 0.$$

The procedure for matching these vectors goes as follows. Pick y^i with $y^i(e) = y^i(f) = 1$, $y^i(g) = 0$ and $\lambda_i > 0$. Pick z^j with $z^j(e) = z^j(f) = 1$, $z^j(g) = 0$ and $\mu_j > 0$. Match these two vectors to obtain χ^p, define $\beta_p = \min\{\lambda_i, \mu_j\}$, set $\lambda_i \leftarrow \lambda_i - \beta_p$, $\mu_j \leftarrow \mu_j - \beta_p$ and continue. $\square$

This theorem shows that one can obtain a description of the polytope if one knows it for wheels. It also shows that the polytope for G is defined by bound and cut inequalities if the polytope for the pieces is defined by this type of constraints. Assume now that the systems of inequalities that define $\mathrm{P}(G_1)$ and $\mathrm{P}(G_2)$ are minimal and that these polytopes are full-dimensional, we are going to prove that the system given in Theorem 2.1 is also minimal.

Theorem 2.2. *Suppose that*

$$ax \geqslant \alpha \tag{2.1}$$

defines a facet $\mathscr{F}$ of $\mathrm{P}(G_1)$ that is not the face defined by $x(e) + x(f) + x(g) \geqslant 2$, then (2.1) *also defines a facet of* $\mathrm{P}(G)$.

Proof. Let $S = \{x_1, \ldots, x_p\}$ be the set of extreme points $\mathrm{P}(G_1)$ that lie in $\mathscr{F}$, then the vector

$$\bar{x} = \frac{1}{p}(x_1 + \cdots + x_p)$$

satisfies $a\bar{x} = \alpha$ and every other inequality in the system that defines $\mathrm{P}(G_1)$ as strict inequality.

Now we have to construct a vector in $\mathrm{P}(G)$ with the same property. Let $T = \{y_1, \ldots, y_q\}$ be the set of extreme points of $\mathrm{P}(G_2)$, since this is a full-dimensional polytope the vector

$$\bar{y} = \frac{1}{q}(y_1 + \cdots + y_q)$$

satisfies all the inequalities that define $\mathrm{P}(G_2)$ as strict inequalities. We say that a vector x_i and a vector y_j are agreeable if they agree in their components associated with e, f and g. Now define a set of extreme points of $\mathrm{P}(G)$ as follows.

Match each vector x_i with an agreeable vector in T to define a vector z_i, $1 \leqslant i \leqslant p$. For each x_i there is an agreeable vector in T, because $P(G_2)$ is full-dimensional.

Match each vector y_i with an agreeable vector in S to define z_{p+i}, $1 \leqslant i \leqslant q$. For each y_i there is an agreeable vector in S, because $P(G_1)$ is full-dimensional and $\mathscr{F}$ is not the face defined by $x(e) + x(f) + x(g) \geqslant 2$.

The vector

$$\bar{z} = \frac{1}{p+q}(z_1 + \cdots + z_{p+q})$$

satisfies (2.1) as equation and every other constraint in the system given by Theorem 2.1 as strict inequality. □

Theorem 2.3. *If the constraint*

$$x(e) + x(f) + x(g) \geqslant 2 \tag{2.2}$$

defines a facet for $\mathrm{P}(G_1)$ *and* $\mathrm{P}(G_2)$ then it also defines a facet for $\mathrm{P}(G)$.

Proof. As in the proof of Theorem 2.2, match vectors in the facet of $\mathrm{P}(G_1)$ with vectors in the facet of $\mathrm{P}(G_2)$ and produce a vector in $\mathrm{P}(G)$ that lies in the face defined by (2.2) and not in any other proper face. □

3. The 2-edge-connected subgraph polytope of a Halin graph

Let $G = (V, T \cup C)$ be a Halin graph. Given a nonpendant node $u \in T$, and $f \in \delta(u)$, set $F_u^f = \bigcup\{\delta_e : e \in \delta(u)\} \backslash \delta_f$. Also set $F_u^0 = \bigcup\{\delta_e : e \in \delta(u)\} \backslash \delta(u)$. The main result of this section is the following.

Theorem 3.1. *Let* $G = (V, T \cup C)$ *be a Halin graph, then a minimal system of inequalities that defines the* TECP(G) *is*:

$x(e) \leqslant 1$ *for every edge* e,

$x(\delta_e) \geqslant 2$ *for every edge* $e \in T$,

$x(\delta(u)) \geqslant 2$ *for every node* $u \notin C$,

$x(F_u^f) \geqslant |\delta(u)| - 1$ *for every nonpendant node* $u \in T$, *and every* $f \in \delta(u)$,

$x(F_u^0) \geqslant \lceil |\delta(u)|/2 \rceil$ *for every nonpendant node* $u \in T$, *with* $|\delta(u)|$ *odd.*

We have seen in the last section that the key is to prove Theorem 3.1 for wheels. Consider a wheel $W_n = (U, F)$ where n is a positive integer $\geqslant 3$. Let $U = \{u_0, u_1, \ldots, u_{n-1}, w\}$ and $F = \{e_0, \ldots, e_{n-1}, f_0, \ldots, f_{n-1}\}$, where $e_i = u_i u_{i+1}$, $f_i = wu_i$, for $i = 0, \ldots, n-1$, throughout this section the indices are taken modulo n. Also denote by C the cycle $\{e_0, \ldots, e_{n-1}\}$. See Fig. 1.

It is easy to see that the following constraints are valid for TECP(W_n):

$$x(F \backslash \delta(u_i)) \geqslant n - 1, \quad i = 0, \ldots, n-1, \tag{3.1}$$

$$x(F \backslash \delta(w)) \geqslant \lceil n/2 \rceil. \tag{3.2}$$

To derive them from (1.3) take $\bar{V} = \{u_0\}$, $V_0^0 = \{w\}$, $V_1^0 = \{u_{n-1}\}$, $V_2^j = \{u_{n-2-j}\}$, $j = 0, \ldots, n-3$; and we obtain an inequality (3.1). When n is odd take $\bar{V} = \{w\}$ and

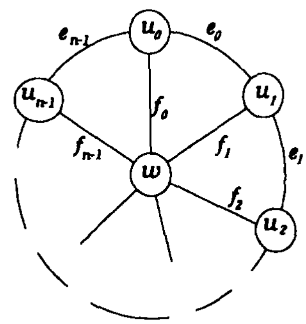

Fig. 1.

$V_i^0 = \{u_i\}$, $i = 0, \ldots, n-1$; and we have (3.2). When n is even, constraint (3.2) can be obtained by adding the cut inequalities associated with the nodes u_i, $0 \leqslant i \leqslant n-1$, and the upper bounds for the edges in $\delta(w)$, so it defines a facet if and only if n is odd. We plan to show that (3.1) and (3.2) together with the bound and cut constraints define TECP(W_n).

Let

$$t(W_n) = \{S \subseteq F \mid (U, S) \text{ is a 2-edge-connected subgraph of } W_n\}.$$

For the valid inequality $ax \geqslant \alpha$ let

$$t_a = \{S \in t(W_n) \mid ax^S = \alpha\}.$$

Suppose that $ax \geqslant \alpha$ defines a facet of TECP(W_n) that is not a bound inequality nor a cut constraint. For any edge e, there is a set $S \in t_a$, with $e \notin S$. Since $S \cup \{e\} \in t(W_n)$, we have that $a \geqslant 0$. Also $\alpha > 0$. Since W_n is 3-edge connected then TECP(W_n) is full dimensional. Thus any valid constraint of TECP(W_n) which is satisfied with equality by every $S \in t_a$ must be a positive multiple of $ax \geqslant \alpha$. We will show that $ax \geqslant \alpha$ is necessarily of type (3.1) or (3.2). In what follows we give a series of lemmas which lead to this result.

Lemma 3.2. *Let $T \in t_a$, if $\delta(u_i) \subseteq T$ for some $i \in \{0, \ldots, n-1\}$ and $a(f_i) > 0$, then $C \subseteq T$.*

Proof. If $C \nsubseteq T$, then since $\delta(u_i) = \{e_{i-1}, e_i, f_i\}$, there are two integers l, p, $0 \leqslant l$, $p \leqslant n-1$, such that $\{e_i, \ldots, e_l\} \subseteq T_i$, $e_{l+1} \notin T$ and $\{e_p, \ldots, e_{i-1}\} \subseteq T$, $e_{p-1} \notin T$, notice that e_{l+1} and e_{p-1} may coincide, recall that the indices are taken modulo n. Thus we have that $f_{l+1}, f_p \in T$, and consequently $T \setminus \{f_i\} \in t(W_n)$, which implies $a(f_i) = 0$, a contradiction. □

Lemma 3.3. *There exists at least one edge in $\delta(w)$ having a zero coefficient in $ax \geqslant \alpha$.*

Proof. Suppose not. Since $ax \geqslant \alpha$ is different from a cut constraint, then for every i, $i = 0, \ldots, n-1$, there is an edge set $T_i \in t_a$ containing $\delta(u_i)$. Otherwise we would have that x^S satisfies $x(\delta(u_i)) = 2$, for every $S \in t_a$.

From Lemma 3.2 and our hypothesis, it follows that $C \subseteq T_i$ for $i = 0, \ldots, n-1$. Thus, the edge sets $(T_i \backslash \{e_i\}) \cup \{f_{i+1}\}$ and $(T_i \backslash \{e_{i-1}\}) \cup \{f_{i-1}\}$ define 2-edge connected subgraphs of W_n, and consequently we have

$$a(e_i) \leqslant a(f_{i+1}), \qquad a(e_{i-1}) \leqslant a(f_{i-1}) \tag{3.3}$$

for $i = 0, \ldots, n-1$.

Furthermore, there must exist an edge set $T_n \in t_a$ such that $|\delta(w) \cap T_n| \geqslant 3$. Since, by hypothesis, $a(f_i) > 0$ for $i = 0, \ldots, n-1$, there must exist two nonconsecutive edges e_j, e_k, $0 \leqslant j < k \leqslant n-1$, which are not in T_n. Thus $\{f_j, f_{j+1}, f_k, f_{k+1}\} \subseteq T_n$. Let $\tilde{T}_n = (T_n \backslash \{f_j, f_{j+1}\}) \cup \{e_j\}$. Clearly $\tilde{T}_n \in t(W_n)$, which implies that $a(e_j) \geqslant a(f_j) + a(f_{j+1})$. This contradicts (3.3). □

Define

$$a(f_i) = \min\{a(f_i)\colon i = 0, \ldots, n-1\} = 0,$$

$$\beta = a(f_l) = \max\{a(f_j)\colon j = 0, \ldots, n-1\},$$

$$\gamma = a(e_m) = \max\{a(e_i)\colon i = 0, \ldots, n-1\}.$$

Lemma 3.4. *$\beta = a(f_j)$ for all $j \neq k$.*

Proof. If $\beta = 0$ there is nothing to prove. So assume that $\beta > 0$. There is $Q \in t_a$ with $\{e_{l-1}, e_l, f_l\} \subset Q$. By Lemma 3.4, $C \subseteq Q$. So $C \cup \{f_l, f_k\} \in t_a$ and $C \cup \{f_j, f_k\} \in t_a$ for each $j \neq l$, k. This shows that $\beta = a(f_j)$ for all $j \neq k$. □

Lemma 3.5. *If $\beta = 0$, then $\gamma = a(e_i)$ for $i = 0, \ldots, n-1$.*

Proof. There is $Q \in t_a$ with $\{e_{m-1}, e_m, f_m\} \subset Q$. If $e_{m+1} \in Q$ then $Q \cup \{f_{m+1}\} \backslash \{e_m\} \in t_a$ implying $a(e_m) = 0$ and $a = 0$, a contradiction. So $\{f_{m+1}, f_{m+2}, e_{m+2}\} \subset Q$. Thus $Q \cup \{e_{m+1}\} \backslash \{e_m\} \in t_a$ implying $\gamma = a(e_m) = a(e_{m+1})$. Now we repeat the same argument and the proof is complete. □

Lemma 3.6. *If $\beta > 0$, then* (i) *$a(e_k) = a(e_{k-1}) = 0$ and* (ii) *$\beta = a(e_j)$ for all $j \neq k, k-1$.*

Proof. Since we are not dealing with a cut inequality, there is $Q \in t_a$ with $\{e_k, e_{k+1}, f_{k+1}\} \subset Q$. By Lemma 3.2 $C \subset Q$. Notice that $C \cup \{f_k, f_{k+1}\}$ and $C \cup \{f_k, f_{k+1}\} \backslash \{e_k\}$ are in t_a, hence $a(e_k) = 0$. By symmetry, $a(e_{k-1}) = 0$.

Now consider (ii). For any $j \neq k$, $k-1$, since we are not dealing with a bound inequality, there exists $Q' \in t_a$ not containing e_j. Note that Q' must contain f_j and f_{j+1}. So $Q' \cup \{e_j, f_k\} \setminus \{f_j\} \in t(W_n)$ implying $a(e_j) \geqslant a(f_j) = \beta$. So $a(e_m) \geqslant \beta$ and $a(f_m) = a(f_{m+1}) = \beta$. Since this is not a cut constraint, there exists $Q'' \in t_a$ containing $\{f_m, e_m, e_{m-1}\}$. By Lemma 3.2, $C \subset Q''$. And $Q'' \cup \{f_{m+1}\} \setminus \{e_m\} \in t(W_n)$ implying $a(f_{m+1}) \geqslant a(e_m)$. Hence (ii) follows. □

Now we are ready to prove our result for wheels.

Theorem 3.7. *A minimal system of inequalities that defines* $\mathrm{TECP}(W_n)$ *is:*

$$x(e) \leqslant 1 \quad \textit{for every edge } e,$$

$$x(\delta(u)) \geqslant 2 \quad \textit{for every node } u,$$

$$x(F \setminus \delta(u_i)) \geqslant n-1, \quad i = 0, \ldots, n-1,$$

$$x(F \setminus \delta(w)) \geqslant \lceil n/2 \rceil \quad \textit{if } n \textit{ is odd.}$$

Proof. Let $ax \geqslant \alpha$ be a facet defining inequality of $\mathrm{TECP}(W_n)$. Suppose that $ax \geqslant \alpha$ is neither a cut constraint nor a bound constraint. By Lemma 3.3, we may assume that $a(f_k) = 0$ for some $k \in \{0, \ldots, n-1\}$. We shall discuss two cases.

Case a: There exists $j \neq k$, such that $a(f_j) = 0$. Thus, from Lemma 3.4 we have $a(f_l) = 0$ for all l. From Lemma 3.5 it follows that $a(e_i) = a(e_j) > 0$ for all $i, j \in \{0, \ldots, n-1\}$. Now it is easy to see that in this case, every edge set $T \in t_a$ must contain exactly $\lceil n/2 \rceil$ edges from $\{e_0, \ldots, e_{n-1}\}$, implying that $ax \geqslant \alpha$ is equivalent to inequality (3.2).

Case b: $a(f_j) > 0$ for all $j \in \{0, \ldots, n-1\} \setminus \{k\}$. From Lemma 3.6 we have that ax is a positive multiple of the left hand side of an inequality in (3.1). It is also easy to see that any set in t_a must contain $n-1$ edges from $F \setminus \{e_k, e_{k-1}, f_k\}$. This implies that $ax \geqslant \alpha$ is a positive multiple of an inequality of type (3.1). □

It is straightforward to see that Theorem 3.1 follows from Theorem 3.7 and the results of Section 2.

4. The 2-node-connected subgraph polytope of a Halin graph

We use here the notation defined in the preceding section. Our main result for the two-node-connected case is the following.

Theorem 4.1. *Let $G = (V, T \cup C)$ be a Halin graph, then a minimal system of inequalities that defines* TNCP(G) is

$$x(e) \leqslant 1 \quad \textit{for every edge } e,$$

$$x(\delta_e) \geqslant 2 \quad \textit{for every edge } e \in T,$$

$$x(\delta(u)) \geqslant 2 \quad \textit{for every node } u \notin C,$$

$$z(F_u^f) \geqslant |\delta(u)| - 1 \quad \textit{for every nonpendant node } u \in T \textit{ and } f \in \delta(u), \tag{4.1}$$

$$x(F_u^0) \geqslant |\delta(u)| - 1 \quad \textit{for every nonpendant node } u \in T. \tag{4.2}$$

Here inequalities (4.1) are of the type (1.1) whereas (4.2) are of the class (1.2).

As seen in Section 2, we have to derive a description of TNCP(W_n). It is easy to see that if G is a graph such that $G \backslash e$ is 2-node-connected for every edge $e \in G$, then TNCP(G) is full dimensional. Notice that TNCP(G) $\subseteq$ TECP(G) so if an inequality is redundant for the second polytope it will also be redundant for the first one. So from Theorem 3.10 we have that the only bound inequalities that are candidates to define facets for a wheel are

$$x(e) \leqslant 1 \quad \text{for every edge } e,$$

and in fact it is not difficult to see that they do define facets. Also from Theorem 3.7 we have that the only cut inequalities that are candidates to define facets are

$$x(\delta(u)) \geqslant 2 \quad \text{for every node } u,$$

and as it is shown in the next two lemmas they do define facets.

Lemma 4.2. *The constraints*

$$x(\delta(u_i)) \geqslant 2 \quad \text{for } i = 0, \ldots, n-1 \tag{4.3}$$

define facets of TNCP(W_n).

Proof. Consider $i = 1$. Denote (4.3) by $ax \geqslant \alpha$ and suppose that $t_a \subseteq t_b$ for a facet defining inequality $bx \geqslant \beta$.

Consider $T = C \backslash \{e_1\} \cup \{f_1, f_2\} \in t_a$. Since $T \cup \{f_i\} \in t_a$ for $i = 0, 3, \ldots, n-1$, we have $b(f_i) = 0$ for $i = 0, 3, \ldots, n-1$. In the same way we can prove that $b(f_2) = 0$.

Consider now $T = C \backslash \{e_2\} \cup \{f_2, f_3\} \in t_a$. Since $T \cup \{e_2\} \in t_a$ we have $b(e_2) = 0$. In the same way we can prove that $b(e_i) = 0$ for $i = 3, \ldots, n-1$.

Since the sets $C \backslash \{e_1\} \cup \{f_1, f_2\}$ and $C \backslash \{e_2\} \cup \{f_2, f_3\}$ are both in t_a we have $b(f_1) = b(e_1)$, in the same way we have $b(f_1) = b(e_0)$. So $a = \lambda b$ and $\alpha = \lambda \beta$, for $\lambda \geqslant 0$. $\square$

Lemma 4.3. *The inequality*

$$x(\delta(w)) \geqslant 2 \tag{4.4}$$

defines a facet of TNCP(W_n).

Proof. Denote (4.4) by $ax \geqslant \alpha$ and suppose that $t_a \subseteq t_b$ for a facet defining inequality $bx \geqslant \beta$.

Consider $T = C\backslash\{e_1\} \cup \{f_1, f_2\} \in t_a$. Since $T \cup \{e_1\} \in t_a$ we have $b(e_1) = 0$. In the same way we can show that $b(e_i) = 0$, for $i = 0, \ldots, n-1$.

Since the sets $C \cup \{f_1, f_2\}$ and $C \cup \{f_2, f_3\}$ are both in t_a we have $b(f_1) = b(f_3)$, in the same way we have $b(f_1) = b(f_i)$ for $i \neq 1$. So $a = \lambda b$ and $\alpha = \lambda\beta$ for $\lambda \geqslant 0$. □

Notice that with the exception of Lemmas 3.3 and 3.5, all proofs of the lemmas of Section 3 are valid for TNCP(W_n). Now we are going to prove the analogues of these two.

Lemma 4.4. *There exists at least one edge in $\delta(w)$ having a zero coefficient in $ax \geqslant \alpha$.*

Proof. Suppose not. Since $ax \geqslant \alpha$ is different from a cut constraint, there must exist an edge set $T \in t_a$ such that $|\delta(w) \cap T| \geqslant 3$.

If $C \subset T$, then for any edge $e \in \delta(w) \cap T$, $T' = T\backslash\{e\}$ is 2-node-connected. Since $ax^{T'} < \alpha$, we have a contradiction.

If there exists e_i such that $C \cap T = C\backslash\{e_i\}$, then $\{f_i, f_{i+1}\} \subset T$. Notice that $T' = C\backslash\{e_i\} \cup \{f_i, f_{i+1}\}$ is two-node-connected, and $T' \subset T$. We would have that $ax^{T'} < \alpha$. A contradiction. □

Lemma 4.5. *If $\beta = 0$, then $\gamma = a(e_i)$ for $i = 0, \ldots, n-1$.*

Proof. Since we are not dealing with a bound inequality, for any e_i there is a $Q \in t_a$ with $e_i \notin Q$. And $C\backslash\{e_i\} \subset Q$, otherwise Q would not be 2-node connected. Let $Q' = Q \cup \{f_j \colon j = 0, \ldots, n-1\}$, we have that $Q' \in t_a$. Consider $Q'' = Q'\backslash\{e_m\} \cup \{e_i\}$. Since Q'' is 2-node connected, $a(e_m) \leqslant a(e_i)$, and then $a(e_m) = a(e_i)$. □

The following theorem gives a description of the polytope for a wheel. Its proof is similar to that of Theorem 3.6.

Theorem 4.6. *The system below defines* TNCP(W_n)

$$x(e) \leqslant 1 \quad \textit{for every edge } e,$$

$$x(\delta(u)) \geqslant 2 \quad \textit{for every node } u,$$

$$x(F\backslash\delta(u)) \geqslant n-1 \quad \textit{for every node } u. \tag{4.5}$$

Actually this is a minimal system, in the next section we shall see that inequalities (4.5) are facet defining. Theorem 4.1 follows from Theorem 4.6 and the results of Section 2.

5. A class of facet defining inequalities for TNCP(G)

Here is a sufficient condition for (1.2) to define a facet of TNCP(G), other conditions for inequalities of this type have been given in [11]. Let $G = (V, E)$ be graph whose node set can be partitioned into $\bar{V}$, V_i^j, $i = 0, \ldots, n-1$, $j = 0, 1, \ldots, p_i$, so that:

(1) each of the members of the partition induces a 3-node-connected subgraph;

(2) there is exactly one edge between V_i^0 and V_{i+1}^0 for $i = 0, \ldots, n-1$ (modulo n);

(3) if $p_i > 0$, there is exactly one edge between V_i^j and V_i^{j+1}, $j = 0, \ldots, p_i$; $i = 0, \ldots, n-1$, where $V_i^{p_i+1} = \bar{V}$;

(4) if V_i^0, $i = 0, \ldots, n-1$, are removed, the only edges that are left are among those described in (3);

(5) there is no edge between V_i^0 and V_i^j, for $j = 2, \ldots, p_i + 1$; $i = 0, \ldots, n-1$,

let

$$F = \bigcup_{\substack{i = 0, \ldots, n-1 \\ j = 0, \ldots, p_i}} \delta(V_i^j) \backslash \delta(\bar{V}).$$

As we shall see later, the partition inequality

$$x(F) \geqslant n - 1 + \sum p_i \tag{5.1}$$

defines a facet of TNCP(G).

Let us see first two examples of this. For a wheel take $\bar{V} = \{w\}$ and $V_i^0 = \{u_i\}$, $i = 0, \ldots, n-1$; and we obtain one of the inequalities (4.3). Now take $\bar{V} = \{u_0\}$, $V_0^0 = \{w\}$, $V_1^0 = \{u_{n-1}\}$, $V_2^j = \{u_{n-2-j}\}$, $j = 0, \ldots, n-3$; and we obtain another inequality (4.3), all inequalities (4.3) can be obtained in this way.

Theorem 5.1. *Given $G = (V, E)$, suppose that $G \backslash e$ is 2-node-connected for every edge e, if G admits a partition that satisfies* (1)–(4) *then inequality* (5.1) *defines a facet of* TNCP(G).

Proof. We denote by $\Theta(G)$ the set of 2-node-connected subgraphs of G. Denote (5.1) by $ax \geqslant \alpha$ and suppose that

$$t_a = \{F \in \Theta(G) \mid ax^F = \alpha\} \subseteq t_b = \{F \in \Theta(G) \mid bx^F = \beta\}$$

for a facet defining inequality $bx \geqslant \beta$. Since TNCP(G) is full dimensional we have to prove that $a = \rho b$ for $\rho > 0$. When appropriate the indices are taken modulo n.

Let us denote by e_{ij} the edge between V_i^j and V_i^{j+1} and set

$$E_0 = \{e_{ij} \mid i = 0, \ldots, n-1;\ j = 0, \ldots, p_i\}.$$

First we have to see that b_e has the same value for all $e \in \delta(V_i^0) \backslash (\{e_{i0}\} \cup \delta(\bar{V}))$, $i = 0, \ldots, n-1$. Let h_i be an edge between V_i^0 and V_{i+1}^0. Consider the edge sets

$$E_1 = \{h_1, \ldots, h_{n-1}\} \cup E_0, \qquad E_2 = (E_1 \backslash \{h_1\}) \cup \{e\},$$

where $e \in \delta(V_1^0) \backslash (\{h_1, e_{1,0}\} \cup \delta(\bar{V}))$. Clearly $\{E_1, E_2\} \subset t_a$, thus

$$0 = bx^{E_1} - bxE^{E_2} = b_{h_1} - b_e.$$

So

$$b_e = \rho \quad \text{for all } e \in \delta(V_i^0) \backslash (\{e_{i0}\} \cup \delta(\bar{V})).$$

By symmetry we obtain

$$b_e = \rho \quad \text{for all } e \in \delta(V_i^0) \backslash (\{e_{i0}\} \cup \delta(\bar{V})), \quad i = 0, \ldots, n-1. \tag{5.2}$$

Next we show that $b_{e_{ij}} = \rho$ for every edge e_{ij}, $j \neq p_i$. Since $G \backslash e_{ij}$ is 2-node-connected, it follows that e_{ij} is not in a 2-edge cutset and by (3) and (4) there is an edge f between V_0^j and some set V_r^0 and there is an edge g between V_i^{j+1} and some set V_s^0. Consider now

$$E_3 = \{h_r, h_{r+1}, \ldots, h_{r+n-1}\} \cup E_0, \qquad E_4 = (E_3 \backslash \{h_r, e_{ij}\}) \cup \{f, g\},$$

since $\{E_3, E_4\} \subset t_a$, we have

$$0 = bx^{E_3} - bx^{E_4} = b_{h_r} + b_{e_{ij}} - b_f - b_g,$$

from (5.2) we obtain $b_f = b_g = b_{h_r} = \rho$, and therefore $b_{e_{ij}} = \rho$.

For every edge e with $a(e) = 0$, there is $Q_e \in t_a$ with $e \notin Q_e$. This is because we are not dealing with a bound inequality. Since $Q_e \cup \{e\} \in t_a$, we have that $b_e = 0$. We have shown that

$$b_e = \rho \quad \text{for all } e \in F, \qquad b_e = 0 \quad \text{for all } e \in E \backslash F.$$

Now consider E_1 and $E_5 = E_1 \cup \{h_0\}$, since $bx^{E_1} = \beta$ and $bx^{E_5} \geqslant \beta$, we have that $\rho > 0$. $\square$

6. Algorithmic aspects

The polyhedral decomposition of Section 2 has an algorithmic analogue. In this section we deal with finding a minimum weighted 2-edge-connected (2-node-connected) subgraph of a Halin graph G. If the present graph is not a wheel then G is decomposed into G_1 and G_2 as before. Let e, f and g be the edges in $G_1 \cap G_2$. Let us denote by $\lambda(S, T, H)$ the minimum weight of a 2-connected subgraph of the graph H, containing the edge set S and having empty intersection with the edge set T.

The edge weights in G_2 are taken to be the same as for G. Then, the problem is solved in G_1 where all the edge weights are taken to be the same as for G, except for $e, f,$

g, which are redefined as the solution of the following system of linear equations:

$$\begin{aligned} w'(e) + w'(f) &= \lambda(\{e,f\},\{g\},G_2) - \kappa, \\ w'(f) + w'(g) &= \lambda(\{f,g\},\{e\},G_2) - \kappa, \\ w'(e) + w'(g) &= \lambda(\{e,g\},\{f\},G_2) - \kappa, \\ w'(e) + w'(f) + w'(g) &= \lambda(\{e,f,g\},\emptyset,G_2) - \kappa. \end{aligned} \tag{6.1}$$

Notice that we had to add the variable κ to guarantee that the system above has a solution. Let $\beta(G)$ be the value of an optimum for G and $\beta(G_1)$ the value of an optimum of G_1 with the new weights. Any solution contains either two or three edges from $\{e,f,g\}$. Since the new weights in G_1 satisfy (6.1), we have that

$$\beta(G) = \beta(G_1) + \kappa.$$

When doing this decomposition we can assume that G_2 is a wheel, it remains to show how to solve the problem in this case.

For the 2-edge-connected case, we use the fact that the complement of a 2-edge connected subgraph of a wheel is a b-matching. More precisely given a wheel $W = (V, T \cup C)$ where T is a star, C is a cycle on the pendant nodes of T, and edge weights $w(\cdot)$, one has to solve

$$\begin{aligned} \text{maximize} \quad & wx \\ \text{subject to} \quad & x(\delta(u)) \leqslant 1 \quad \text{for every node } u \in C, \\ & x(\delta(v)) \leqslant |\delta(v)| - 2 \quad \text{for the center } v, \\ & x \in \{0,1\}^{|T \cup C|}. \end{aligned}$$

This can be solves as a matching problem with Edmonds' algorithm [8].

The 2-node-connected case is easier. For a wheel W_n, one has to enumerate n Hamilton cycles. Then one should add all edges with negative weight, if there is any.

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